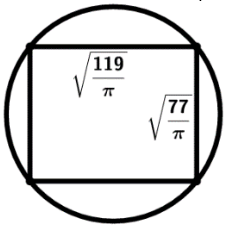
Workshop questions and solutions

E: Sprint 17.

A rectangle with sides $\sqrt{\frac{119}{\pi}}$ and $\sqrt{\frac{77}{\pi}}$ is inscribed in a circle. What is the area of the circle?



Use the Pythagorean theorem for a right triangle $a^2 + b^2 = c^2$.
The diameter of the circle, d, satisfies:

$$d^2 = \frac{119}{\pi} + \frac{77}{\pi} = \frac{196}{\pi}.$$

The radius of the circle, r, is $\frac{d}{2}$.
Thus, the area of the circle is:

$$\pi r^2 = \frac{\pi d^2}{4} = \frac{\pi 49}{\pi} = 49.$$

M: Sprint 23.
In how many ways can you pay 80 cents using any combination of 5, 10, and 25 cent coins?

Divide the possibilities of the sum of 80 cents into the following groups.
a) 3 0.25$, b) 2 0.25$, c) 1 0.25$, or d) 0 0.25$.

Check each of the groups for the number of options of 0.10$.
a) 0.75$ in 0.25$ coins: only 1 option for 0.10$: 0 coins of 0.10$.
b) 0.50$ in 0.25$ coins: 4 options for 0.10$: $0,1,2,$ or 3.
c) 0.25$ in 0.25$ coins: 6 options: $0,1,2,3,4,$ or 5.
d) 0.00$ in 0.25$ coins: 9 options: $0,1,2,3,4,5,6,7,$ or 8.

Total number of ways is:

$$1 + 4 + 6 + 9 = 20.$$

H: Sprint 24.

$N+(N+1)+(N+2)+\cdots+2023=94\times 1000$, and $N>0$. What is the Value of N?

Note that $N>0$. An easy way to solve is simply by observing that 94000 is sum of 47 numbers around the value of 2000 each. Thus, a simple way to solve will be to simply take such 47 consecutive numbers and check. If a student does not see this right away, then some sweating is required.

Use the summation formula of consecutive integers:

$N+(N+1)+(N+2)+\cdots+2023=(N+2023)(2024-N)/2$.

So, from the equation above, $\frac{(N+2023)(2024-N)}{2}=94\times 1000=47\times 2000$.

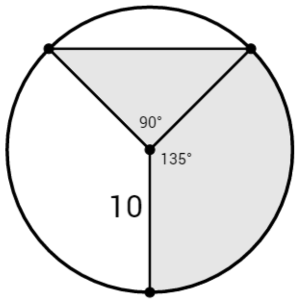
Thus, $(N+2023)(2024-N)=47\times 4000.$

Note: in secondary school students learn that this equation, (quadratic equation), has either 0, 1, or 2 solutions for the unknown N. We do not expect students in elementary school to know how to solve this equation using the quadratic formula. But, from the way the question is phrased, the sum is of is of consecutive integers, and thus N is either a positive whole number or a negative integer. Thus, we need to assume that there is a positive whole number solution to the above equation. 47 is a prime number, so, for N positive, it must be a factor of exactly one term of the multiplication above, (assuming that both terms are positive).
It is now the time to make an educated guess based on the information above.
So, try and plug $2024-N=47$.
Thus, $N=1977$.

It is immediately clear, based on this guess, that $N+2023=4000$, so, $N=1977$ is a solution. As mentioned above.
As mentioned above, the quadratic equation, indeed, has another solution. But the other solution is negative: $N=-1976$.

E: Target 4.

The shaded section of the circle of radius 10 consists of a right triangle and a sector of $135°$. Find the area of the shaded section. Use $\pi = 3.14$, and round your answer to the nearest whole number.



The shaded area is consisted of a right triangle and a sector of the circle.
The area of the right triangle is:
$\frac{10\times10}{2} = 50$.
The area of the $135°$ sector is (using the approximation):
$3.14 \times 10^2 \times \frac{135}{360} = 314 \times \frac{3}{8} = 117.75$.
So, the area of shaded section is:
$50 + 117.75 = 167.75$.
Rounding the area to the nearest whole number the requested solution is: 168.

M: Target 10.

N is the smallest positive whole number such that all the following conditions are satisfied:
$\{a, b, c, d, e, f\}$ is a set of 6 different primes, $N = a + b + c = d + e + f$, $a < b < c$, and $d < e < f$. What is the maximum possible value of $c - a$?

Note that N must be odd based on the condition of the value of N and that the sum of the 6 different numbers of the set is $2N$. Also, since N is odd then the only even prime number, 2, is not a part of the set.

Set of the smallest, odd primes is the set:
$\{3,5,7,11,13,17\}$.
But, $3 + 5 + 7 + 11 + 13 + 17 = 56 = 2 \times 28$. So, this set does not satisfy the condition as N must be odd.

Next set of odd primes with smallest sum is the set:
$\{3,5,7,11,13,19\}$.
Thus, $3 + 5 + 7 + 11 + 13 + 19 = 58 = 2 \times 29$.
The only way to divide this set into 2 groups of 3 members each with equal sum of all members of each group is:
$\{3,7,19\}, \{5,11,13\}$, as $29 = 3 + 7 + 19 = 5 + 11 + 13$.
Thus, maximum possible value of $c - a$ is:
$c - a = 19 - 3 = 16$.

H: Sprint 26.

Frank wrote down the sum of the digits of every number from 1 to 1000. (Examples: for the number 2 he wrote 2, for the number 24 he wrote 6 because $2+4=6$, for the number 200 he wrote 2 because $2+0+0=2$, and for the number 550 he wrote 10 because $5+5+0=10$). How many times did he write the digit 0?

Consider all possible cases where the digit sum of a positive number contains the digit 0 .

The number with the largest possible digit sum is: 999, because $9+9+9=27$.

Thus, the only possible candidate sums that contain the digit 0 are the digit sums:

A) 10, and B) 20 .

Case A: Sum is 10

For 3-digit numbers,

No xxx and no x00 where the digit x is not 0.

For xx0 (or x0x), $x+x+0=10$, we have the numbers 550 and 505, so **2** times the digit 0 is written.

For xyo, yx0, x0y, y0x, x and y are different digits and none is 0, we have the following possible sums (4 times for each selection):

$9+1+0, 8+2+0, 7+3+0, 6+4+0$. So total number that the digit 0 is written is **16**.

For xyy, 3 times for each of: $8+1+1, 6+2+2, 4+3+3, 2+4+4$. So total is **12.**

For xyz, 6 times for each of: $7+2+1, 6+3+1, 5+4+1, 5+3+2$. So total is **24.**

For 2-digit numbers,

No x0

For xx, 1 time for $5+5$. Total is **1.**

For xy, 2 times for each of: $9+1, 8+2, 7+3, 6+4$. Total is **8.**

Total for digit sum of 10: $2+16+12+24+1+8=63$.

Case B: sum is 20

No xxx

No xx0

No x00

No xy0

For xyy, 3 times for each of $8+6+6, 6+7+7, 4+8+8, 2+9+9$. Total is **12.**

For xyz, 6 times for each of $9+8+3, 9+7+4, 9+6+5, 8+7+5$. Total is **24.**

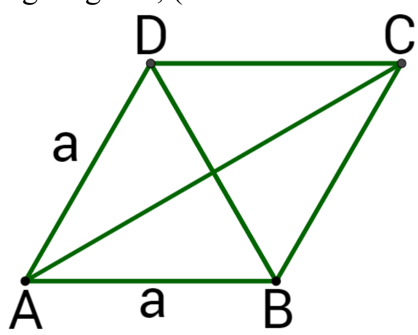
No x0, xx, xy, x

Total for digit sum of 20: $12+24=36$

Thus, $63+36=99$

E: Target 5.

Below is a rhombus with sides $a = 7$. Its short diagonal satisfies $BD = a = 7$. What is the square value of the long diagonal, (i.e. the value of AC^2) ?



In a rhombus BD is perpendicular to AC. Also, the 2 diagonals are bisected.

Using the Pythagorean Theorem $(\frac{AC}{2})^2 + (\frac{a}{2})^2 = a^2$.

So, $(\frac{AC}{2})^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3}{4}a^2 = \frac{3}{4} \times 49 = \frac{147}{4}$.

So, $AC^2 = 4 \times \frac{147}{4} = 147$.

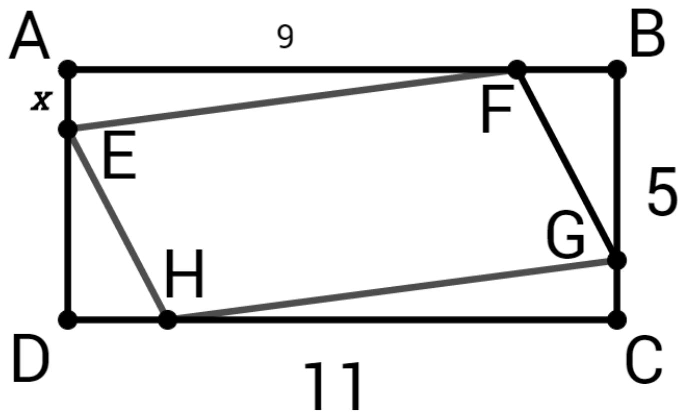
M: Sprint 13.

Dan read a 650 page book in the following way. On the first day he read every second page of the book starting at page 1 (i.e. he read pages 1, 3, 5, and so on). On the second day he read every third page of the book starting at page 1 (i.e. he read pages 1, 4, 7, and so on). How many of the pages did he read twice?

Dan read twice every 6-th page starting at page 1. So, he read twice pages $1,7,13,\cdots,649$. So he read twice: $\frac{654}{6} = 109$ pages.

H: Sprint 25.

$ABCD$ is a rectangle with sides 11 and 5. $EFGH$ is a parallelogram. $AE = x, AF = 9$. The area of $EFGH$ is $\frac{2}{3}$ of the area of $ABCD$. What is the area of ΔDEH? Express the answer as a fraction in lowest terms.



$EFGH$ is a parallelogram. Note that EF is not necessarily perpendicular to EH.

Area of $ABCD$ is 55 so the sum of the areas of AEF, EDH, HCG, and GBF is $\frac{55}{3}$.

$FB = AB - AF = 11 - 9 = 2$.

Thus, $9x + 2(5 - x) = \frac{55}{3}$.

$9x + 10 - 2x = \frac{55}{3}$

Multiply by 3: $21x + 30 = 55$, so 21x=55-30=25.

Thus, $x = \frac{25}{21}$.

Area of triangle DEH is $\frac{2(5-x)}{2} = 5 - x = 5 - \frac{25}{21} = \frac{105-25}{21} = \frac{80}{21}$

E: Sprint 19.

$\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = 3$. Express x as a fraction in lowest terms.

$6x$ is a common denominator so $\frac{6+3+2}{6x} = 3$.

So, $11 = 6x \times 3 = 18x$, so $x = \frac{11}{18}$.

M: Target 8.

What is the sum of all factors of 2023? (**Hint**: 2023 is not a prime number).

2100 is divisible by 7, so 2030 is divisible 7, so 2023 is divisible by 7.

Thus, $\frac{2023}{7} = 289 = 17^2$, so, $2023 = 17^2 \times 7$.

Using the formula for the sum of all factors we get:

$(1 + 17^1 + 17^2) \times (1 + 7^1) = (1 + 17 + 289) \times (1 + 7) = 307 \times 8 = 2456$.

H: Target 12.

A triangle has sides L, M, and N, where $0 < L < M < N < 12$ are all whole numbers. The perimeter of the triangle is P. How many different values of P are there?

For any triangle with sides X, Y, and Z: $X + Y > Z$:

So, the triangle with whole numbers that satisfy the requirements, $(L = 1, M, N)$ is not possible.

So the minimum possible value of P is $P = 9$ of a triangle (L=2,M=3,N=4).

$P = 10$ is not possible: $(L = 2, M = 3, N = 5)$ is not a triangle because $2 + 3 = 5$. Also not acceptable are: $(L = 2, M = 4, N = 4)$, and $(L = 3, M = 3, N = 4)$, ($L = M$ or $M = N$ is not acceptable).

$P = 11$ ok, $(2,4,5)$.

$P = 12$ ok, $(3.4.5)$.

$P = 13$ ok, $(3,4,6)$.

$P = 14$ ok, $(3,5,6)$.

$P = 15$ ok, $(3,5,7)$ and more.

And, so on, …

Max perimeter is $P = 30$, $(9,10,11)$.

Summary of P values:

$\{9,11,12,13, \cdots ,30\}$. So total is 21 different values of P.

H: Target 3.

There is a pile of 7 cards numbered $1,2,3, \cdots ,7$ on the table. Gloria takes 3 different cards at random from the pile and writes down the sum of these 3 cards. What is the probability that the sum is a multiple of 3? Express the answer as a fraction in lowest terms.

As shown in many workshop before, there is a formula for the number of ways to select 3 items out of 7 items: $\frac{7!}{3!4!} = \frac{5\times6\times7}{2\times3} = 35$.

Systematically it can easily be shown that there are 13 possible ways for the sum to be a multiple of 3:

$\{123,126,135,147,156,234,237,246,267,345,357,456,567\}$.

So, the probability is $\frac{13}{35}$.

H: Sprint 22.
Eric takes 4 times longer to paint a ceiling than to paint a wall. He charges 20% more per hour to paint a ceiling than to paint a wall. Eric painted 5 ceilings and 12 walls. His hourly charge per wall was $40 and his total earning was $2520. How many hours did he work in total?

Define T to be the time in hours to paint a wall.
Thus, $4T$ is the time in hours to paint a ceiling.
$40T$ is the earning in $ to paint a wall.
Thus, $4 \times 1.2 \times 40T = 192T$ is the earning in $ to paint a ceiling.
Thus, for 12 walls and 5 ceilings the earning (in $) is:
$2520 = 12 \times 40T + 5 \times 192T = (480 + 960)T = 1440T$.
Thus, $T = \frac{2520}{1440} = 1.75$ is the time in hours to paint a wall.
Total time (in hours) is then: $1.75 \times 12 + 1.75 \times 5 \times 4 = 21 + 35 = 56$.
Check the solution: $21 \times 40 + 35 \times 48 = 840 + 1680 = 2520$.

M: Target 11.
Jill drives a fuel-efficient car that consumes, on average, 6 litres of fuel per hour. When she started driving, the fuel tank was full. After driving T hours she stopped and added 10 litres of fuel so that the tank was 85% full. Then, she drove $\frac{T}{2}$ hours, stopped again, and filled the tank with 17 more litres of fuel so that the fuel tank was full again. How many liters of fuel can a full tank hold? Round your answer to the nearest whole number.

In T hours fuel consumption was $6T$ litres. In $\frac{T}{2}$ hours the fuel consumption was $3T$ litres.
Thus, total fuel during the entire drive was $6T + 3T = 9T$ litres.
Total fuel added was $10 + 17 = 27$ litres.
So, $9T = 27$, and thus $T = 3$ hours.
Define V to be the volume of the fuel tank (in litres).
Thus, $0.85V = V - 6T + 10 = V - 18 + 10 = V - 8$.
Thus, $0.15V = 8$, so $V = \frac{8}{0.15} = \frac{800}{15} = \frac{160}{3} = 53.333\cdots$ litres.
Rounding to the nearest whole number: $V = 53$.

M: Sprint 3.
The parliament proposes an increase of 400% to the current carbon tax to a new tax rate of $120 per tonne. What is the current tax rate per tonne (in $)?

Increasing by 400% is the same as multiplying by 5.
Thus, the current tax rate is: $\frac{120}{5}$=24.

E: Sprint 15.
For every 1400 boys members of the Canadian Arts and Science Club there are 2023 girls that are members of the club. What percentage of the club membership are the girls? Round your answer to the nearest whole number.

Percentage of girls is given by:

$$100 \times \frac{2023}{2023+1400} = 100X\frac{7\times 289}{7\times(289+200)} = 100X0.591\cdots = 59.1\cdots.$$

So, the percentage of girls is 59.

E: Target 7.
Two teams, A and B, compete in a basketball championship. The probability of Team A to win a game is 80%, and the probability of Team B to win a game is 20%, (no ties). The first team to win 3 games in total wins the championship. What is the probability that it will take only 3 games to decide the championship? Express the answer as a fraction in lowest terms.

Either team A wins 3 times in a row, or team B wins 3 times in a raw.
So, the probability is:

$$\left(\frac{20}{100}\right)^3 + \left(\frac{80}{100}\right)^3 = \left(\frac{1}{5}\right)^3 + \left(\frac{4}{5}\right)^3 = \frac{1}{125} + \frac{64}{125} = \frac{65}{125} = \frac{13}{25}.$$