Introduction to Problem Solving

Four Principles

- *How to Solve It* suggests the following steps when solving a Mathematical Problem:

- Understand the problem
- Make a plan
- Carry out the plan
- Look back on your work. How could it be better?

Danny raises some chickens and rabbits in this little farm. These animals have 15 heads and 40 feet altogether. How many chickens and rabbits does he raise?

- Step One: Understand
- How many chickens and rabbits are there altogether?
- How many feet are there altogether?
- How many feet does each chicken have?
- How many feet does each rabbit have?

Second principle: Devise a plan

- 1. Use Diagrams / Models
  2. Act it Out
  3. Use Before & After
  4. Use Systematic Listing
  5. Look for Patterns
  6. Work Backwards
  7. Use Guess & Check
  8. Simplify the Problem
  9. Make Supposition
  10. Solve Part of the Problem
  11. Paraphrase the Problem

- Use of these models to solve problems

2/5 of a number equals 12. What is the number?

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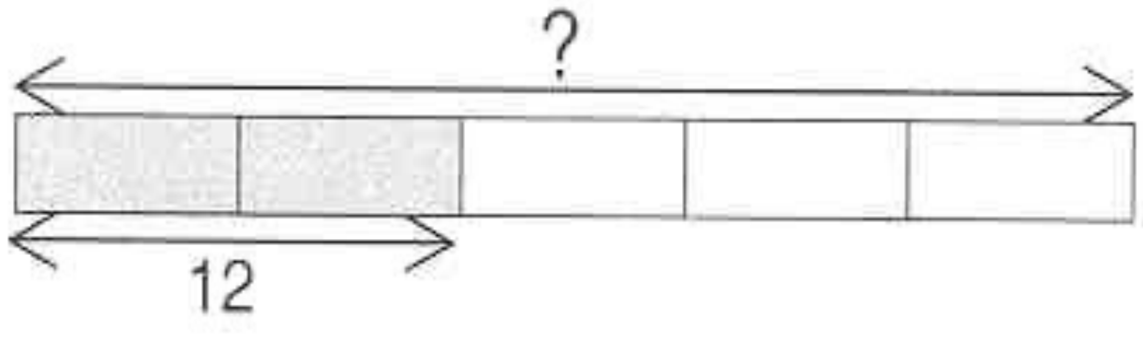

2 units ——> 12
1 unit ——> $12 \div 2 = 6$
5 units ——> $5 \times 6 = 30$

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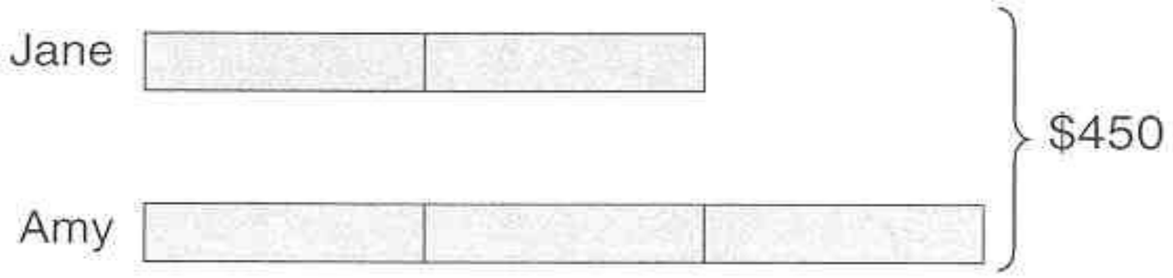

5 units ——> $450

1 unit ——> $\$450 \div 5 = \$90$

2 units ——> $2 \times \$90 = \$180$

The ratio of Simon’s to Ramon’s marbles is 3:5.
Simon has 42 marbles. Simon buys 8 marbles more.
Find the new ratio of Simons’ to Ramon’s marbles.

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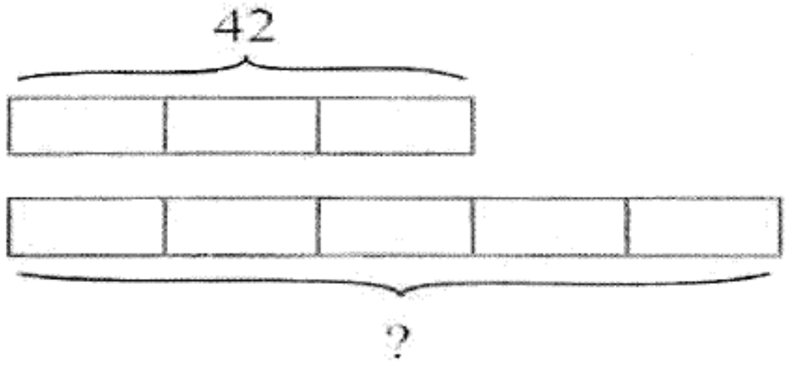
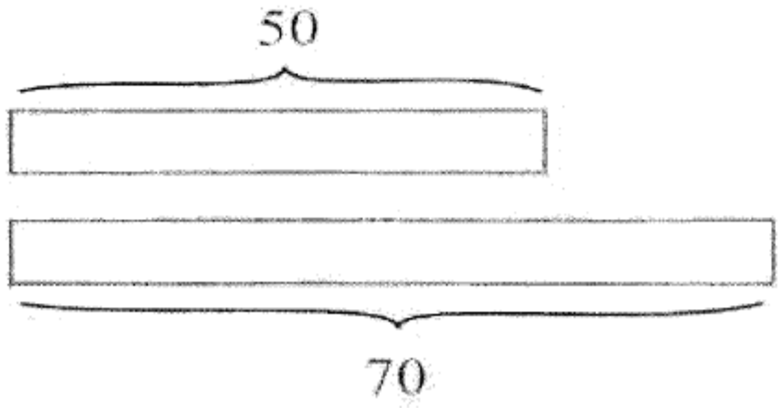


3 units = 42
1 unit = 14
5 units = 70



After:
Simon has 50 marbles.
Ramon has 70 marbles.
50:70 = 5:7

Sam has 3/7 the amount of marbles that Lisa has. Sam gives Lisa 1/6 of his marbles. What will be the new ratio between the number of marbles of Sam and Lisa?

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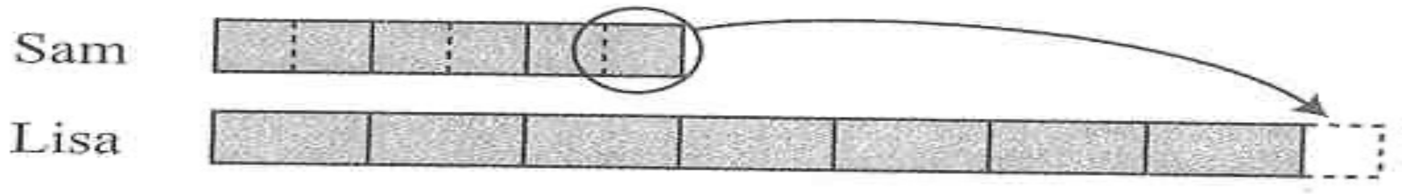
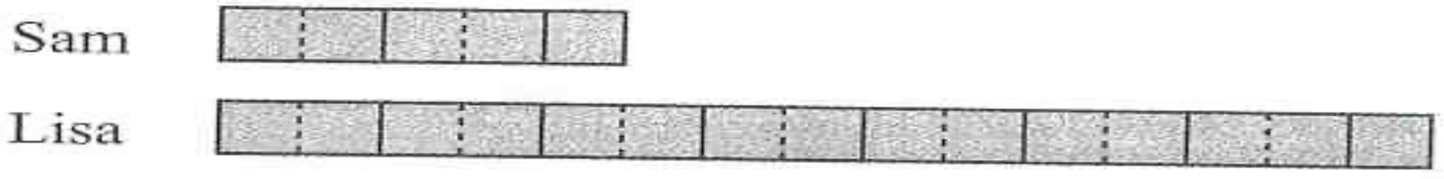
Before:



After



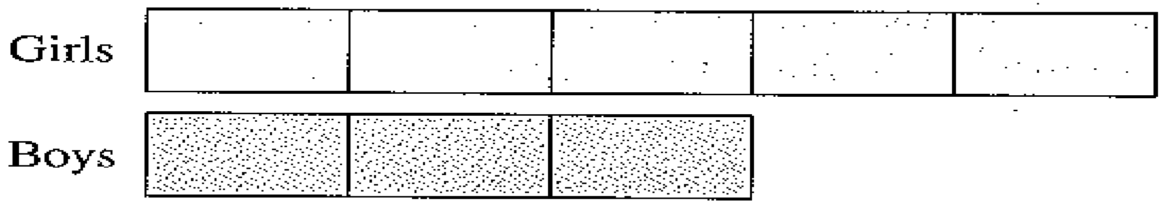
The new ratio of Sam's to Lisa's marbles is 5:15 or 1:3

- John is 15 kgs heavier than Samuel who weighs twice as much as Paul. If the three together weigh 125 kgs, how much does John weigh?

- The ratio of Sam’s tickets to Henry’s started as 7:3. After Sam gave Henry 14 of his tickets, both boys had the same number of tickets. How many tickets did Henry start with?

- The fruit bowl contains red and green grapes. Initially, two thirds (2/3) of the grapes are green. After two sevenths (2/7) of the green grapes are eaten, there are 20 green grapes left. How many of the grapes are red?

The following comparison model shows that there are $\frac{3}{5}$ as many boys as girls:



Difference	The difference between the number of girls and the number of boys is 2 units.
Fraction	The number of boys is $\frac{3}{5}$ times the number of girls.
	The number of girls is $\frac{5}{3}$ times the number of boys.
Ratio	The ratio of the number of boys to the number of girls is 3 : 5.
	The ratio of the number of girls to the number of boys is 5 : 3.

If A is 80% of B, then:

$$A = 0.8B \qquad \left(\text{or} \quad A = \frac{4}{5}B\right)$$

$$B = \frac{A}{0.8} = 1.25A \qquad \left(\text{or} \quad B = \frac{5}{4}A\right)$$

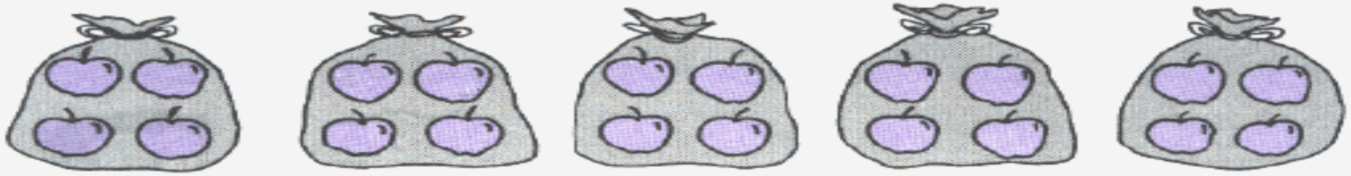
B is 125% of A.

Percentage, fraction and decimal are related. For example,

Percentage	A is 80% as much as B.	B is 125% as much as A.
	A is 20% less than B.	B is 25% more than A.
Fraction	A is $\frac{4}{5}$ times B.	B is $\frac{5}{4}$ times A.
Decimal	A is 0.8 times B.	B is 1.25 times A.

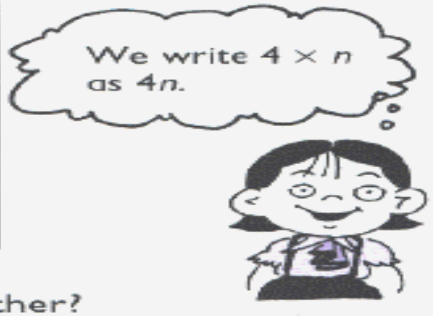
Algebra

4. There are 4 apples in each packet.



(a) How many apples are there in n packets?

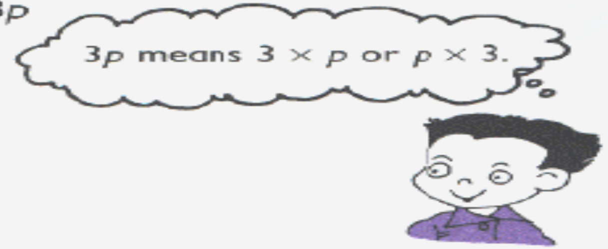
Number of packets	Total number of apples
1	$4 \times 1 = 4$
2	$4 \times 2 = 8$
3	$4 \times 3 = 12$
4	$4 \times 4 = 16$
5	$4 \times 5 = 20$
n	$4n$



(b) If $n = 8$, how many apples are there altogether?
(c) If $n = 11$, how many apples are there altogether?

5. There are 3 boxes of chicken wings. Each box contains p chicken wings.
(a) Express the total number of chicken wings in terms of p.

Total number of chicken wings $= 3p$



(b) If each box contains 7 chicken wings, how many chicken wings are there altogether?

$3p = 3 \times 7 = \blacksquare$

6a

There are $\blacksquare$ chicken wings altogether.

Math Language

- Sum Increased Difference
- Twice Less than More than
- Times Product Times the sum
- A number Is

Examples:

Five less than three time the number:

Twice the sum of a number and five:

Five increased by four times a number:

Twice a number decreased by ten:

Math Language

Examples:

Five less than three time the number: *3n -5*

Twice the sum of a number and five: *2(n+5)*

Five increased by four times a number: 5 + 4n

Twice a number decreased by ten: *2n – 10*

The cost of a rental car is $35 plus 15 cents per mile. Express the cost of renting a car in terms of numbers of miles driven.

C= total cost

m= miles driven

$C = 35 + 0.15m$

- This year I am 9 times older than my grandson. Next year I will be 8 times older than my grandson.

This year, how old are my grandson and I?

- The lengths of the sides of a quadrilateral are four consecutive whole numbers. If its perimeter is 86 inches, find the length of the shortest side.

Integration of the Model Method and Algebra

For the following example, draw the model, obtain the equation, and solve the equation. You can do this in several variations, depending on what you declare as the variable..

- Angelica had \$250. She bought 8 shirts at \$X each.
    - Express the amount of money she had left in terms of X.
    - If X = 10, how much money did she have left?

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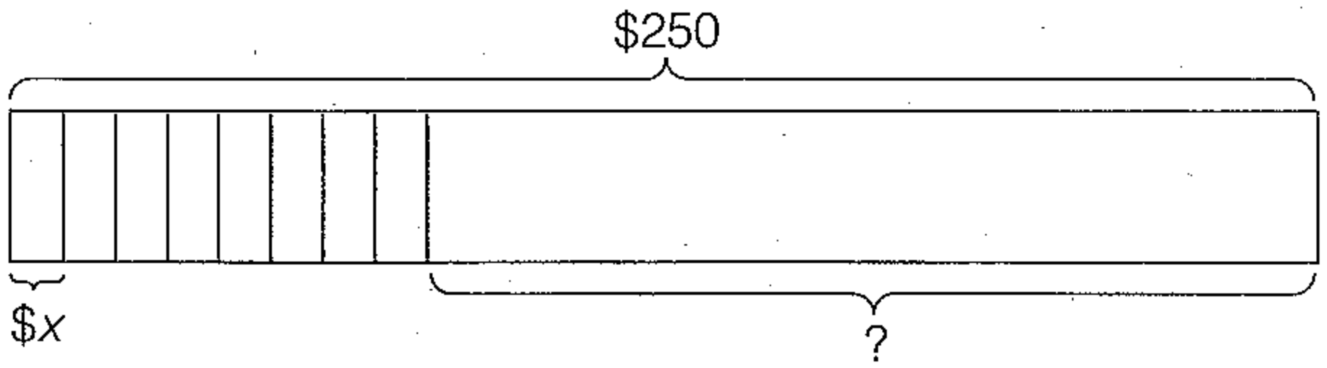

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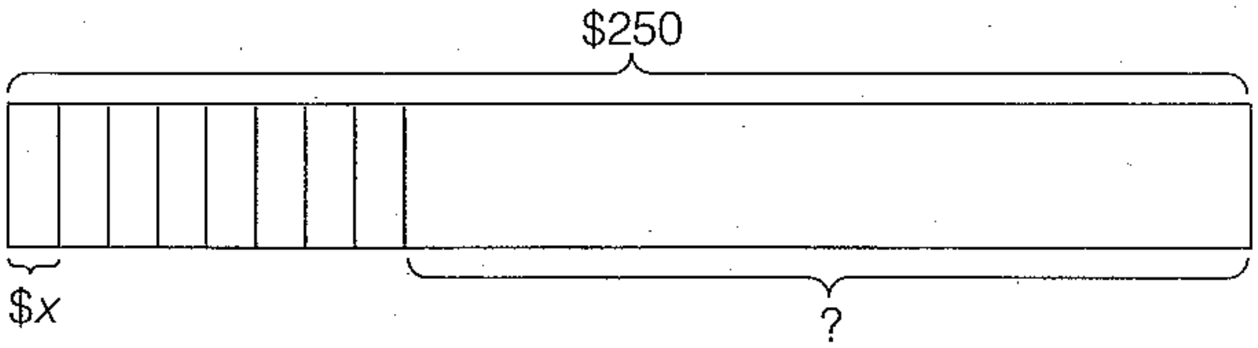

(a) Total cost of the 8 shirts = $8 \times x$ = \$$8x$

Amount of money Angelica had left = \$250 – \$$8x$

= \$($250 - 8x$)

She had **\$(250 – 8*x*)** left.

Angelica had \$250. She bought 8 shirts at \$X each.

(a) Express the amount of money she had left in terms of X.

(b) If X = 10, how much money did she have left?

(b) If $x = 10$, then

$$\begin{aligned} 250 - 8x &= 250 - (8 \times 10) \\ &= 250 - 80 \\ &= 170 \end{aligned}$$

She had **\$170** left.

- Jeffrey had Y cookies. He ate 8 cookies and shared the remaining cookies among his 6 cousins equally.
  - How many cookies did each cousin receive? Express your answer in terms of Y
  - If y= 80, how many cookies did each cousin receive

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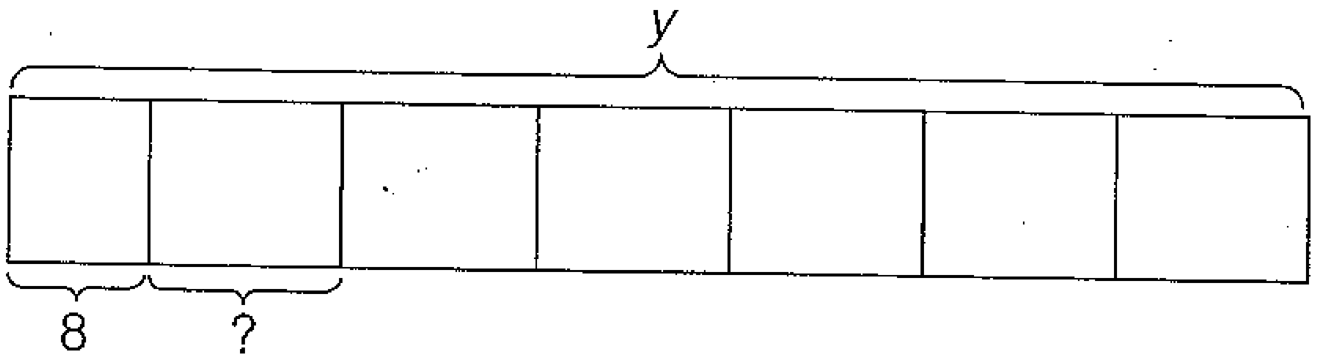

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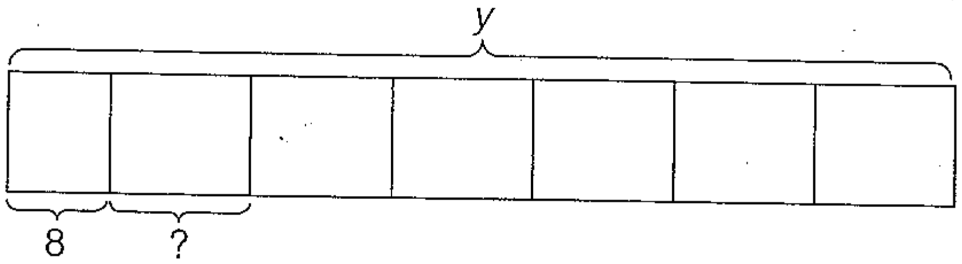

(a) After eating 8 cookies, Jeffrey had $(y - 8)$ cookies left.

$$(y - 8) \div 6 = \frac{y - 8}{6}$$

Each cousin received $\frac{\boldsymbol{y - 8}}{\mathbf{6}}$ cookies.

Jeffrey had Y cookies. He ate 8 cookies and shared the remaining cookies among his 6 cousins equally.
How many cookies did each cousin receive? Express your answer in terms of Y
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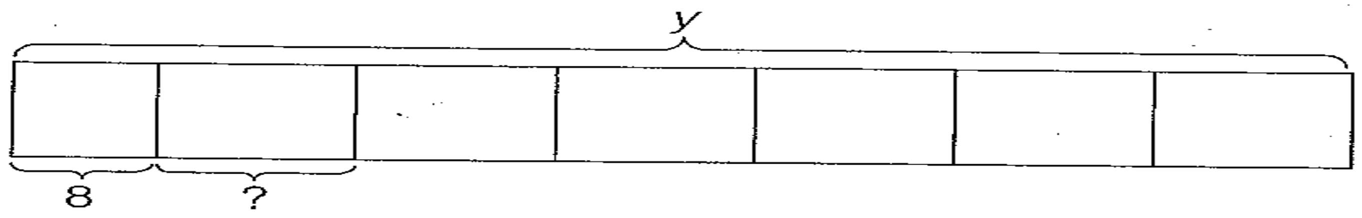

(a) After eating 8 cookies, Jeffrey had $(y - 8)$ cookies left.

$$(y - 8) \div 6 = \frac{y - 8}{6}$$

Each cousin received $\frac{y - 8}{6}$ cookies.

(b) If $y = 80$, then $\frac{y - 8}{6} = \frac{80 - 8}{6}$

$$= \frac{72}{6}$$
$$= 12$$

Each cousin received **12 cookies**.

What is an arithmetic sequence?

- An **arithmetic sequence** is an ordered set of numbers that have a **common difference** between each consecutive term.

  For example in the arithmetic sequence 3, 9, 15, 21, 27, the common difference is 6.

  An arithmetic sequence can be known as an arithmetic progression. The difference between consecutive terms is an arithmetic sequence is always the same.
- If we **add** or **subtract** by the **same number** each time to make the sequence, it is an **arithmetic sequence**.
- The **term-to-term rule** tells us how we get from one term to the next.
- Here are some examples of arithmetic sequences:

Here are some examples of arithmetic sequences:

First Term	Term-to-Term Rule	First 5 Terms
3	Add 6	3, 9, 15, 21, 27, …
8	Subtract 2	8, 6, 4, 2, 0, …
12	Add 7	12, 19, 26, 33, 40, …
-4	Subtract 5	-4, -9, -14, -19, -24, …
½	Add ½	½, 1, 1½, 2, 2½, …

Arithmetic Sequence formula:

$a_n = a_1 + (n-1)d$

$$3,\ 9,\ 15,\ 21,\ 27, \ldots$$

$$\boxed{+6}\ \boxed{+6}\ \boxed{+6}\ \boxed{+6}$$

$$
\begin{aligned}
a_1 &= 3 \\
a_2 &= 3 + 6 \\
a_3 &= 3 + (\underline{2} \times 6) \\
a_4 &= 3 + (\underline{3} \times 6) \\
a_5 &= 3 + (\underline{4} \times 6) \\
&\vdots \\
a_n &= a_1 + (\underline{n-1})d
\end{aligned}
$$

Examples:

- Calculate the next three terms for the sequence 4, 7, 10, 13, 16, …
- Calculate the next three terms for the sequence -3, -9, -15, -21, -27, …
- Calculate the next 3 terms of the sequence 5, 3, 1, −1, −3,…5
- By finding the common difference, state the next 3 terms of the sequence −37,−31,−25,−19,−13, …

More examples:

- Calculate the sum of the 1st ,10th, 100th ,and 1000th term of the sequence $4n-25$

An arithmetic sequence means you are adding some number to each term to get to the next term.

Find d and a1

- Can you see that to go from a9 to a14, we need to add "d" five times therefore:
- 30 + 5d = -15
- 5d = -15 - 30
- 5d = -45
- d = -45/5
- d = -9

- Using the equation for an arithmetic sequence: $a_n = a_1 + d(n - 1)$ we can use the fact that $a_9 = 30$ and plug the values of n = 9 and $a_9 = 30$ into the equation:
- $30 = a_1 + (-9)(9 - 1)$
- $30 = a_1 + (-9)(8)$
- $30 = a_1 + -72$
- $30 + 72 = a_1$
- $a_1 = 102$

Percent means per 100, or divided by 100. Dividing by 100 moves the decimal point two places to the left.

$$24\% = \frac{24}{100} = .24$$

•To convert a fraction or decimal to a percentage, multiply by 100:

Multiply the fraction by 100 to give the result as a percentage value.

$$\frac{1}{5} \times 100 = 5\overline{)100}^{\;20}$$

•To convert a percent to a fraction, divide by 100 and reduce the fraction (if possible):

Divide the percentage value by 100 and simplify the fraction if necessary.

$$60\% = \frac{60}{100} = \frac{3}{5}$$

•12 people out of a total of 25 were female. What percentage were female?

Multiply by 100. Dividing the top and bottom by 25 (cancelling) leaves 12 x 4.

$$\frac{12 \times \cancel{100}^{4}}{\cancel{25}} = 48\%$$

•The price of a $1.50 candy bar is increased by 20%. What was the new price?

Multiply the price by 20% (20/100). Add the result to the original price. ($1.50 + .30 = $1.80)

$$\$1.50 \times \frac{20}{100} = \$0.30$$

$$\$1.50 + \$0.30 = \$1.80$$

- The tax on an item is $6.00. The tax rate is 15%. What is the price without tax?

The price, p, times 15% (15/100) equals 6. Solve the equation by multiplying both sides by 100 and then dividing both sides by 15. The price without tax (P) is 40.

$$P \times \frac{15}{100} = 6$$

$$P \times \frac{15}{\cancel{100}} \times \cancel{100} = 6 \times 100$$

$$P \times \frac{\cancel{15}}{\cancel{15}} = \frac{600}{15} = 40$$

- If a car dealership gives a 5% discount on a car, the dealership will make a $5250 profit on the car. If, instead it will give a 25% discount, the dealership will lose $1750. How much did the dealership pay for the car (in dollars)?
-
- Gavin bought a pair of running shoes on sale for $98. This was 30% less than the usual price. What was the usual price?
-
- 10 points) If a price is reduced by 20% and then increased by 20%, what percent is the overall change?

A rate of work problem will be one in which two (or maybe more) objects or people perform a certain task at different rates. You are asked to find how long it will take them to do a job/task together, or maybe how long an individual would take if working alone.

- **Formula for Rate of Work Problems**

$$\frac{1}{T_A} + \frac{1}{T_B} = \frac{1}{T_T}$$

Where T_A is the individual time for object A

T_B is the individual time for object B

and T_T is the time for A and B together.

Painter A can paint the Smith's house in 12 hours. Painter B could paint the same house in only 9 hours. How many hours would it take to paint the Smith's house if the two painters worked together?

$$T_A = 12\ hours$$

$$T_B = 9\ hours$$

$$T_T = x\ hours$$

$\boxed{\frac{1}{T_A} + \frac{1}{T_B} = \frac{1}{T_T}}$	Formula
$\frac{1}{12} + \frac{1}{5} = \frac{1}{x}$	Input values
$\left(\frac{3}{3}\right)\left(\frac{1}{12}\right) + \left(\frac{4}{4}\right)\left(\frac{1}{9}\right) = \frac{1}{x}$	Obtain common denominator
$\frac{3}{36} + \frac{4}{36} = \frac{1}{x}$	Add
$\frac{7}{36} = \frac{1}{x}$	Cross multiply
$7x = 36$	Solve for x
$x = \frac{36}{7} = \boxed{5.14\ hours}$	

- A computer can process some data in 3 hours.
- If it works together with another computer in the office, they can process the same data in only 1 hour.
- How long would the second computer take to process this data if it didn't work with the first computer?

$$\boxed{\begin{aligned} T_A &= 3\ hours \\ T_B &= x\ hours \\ T_T &= 1\ hours \end{aligned}}$$

$$\boxed{\frac{1}{T_A} + \frac{1}{T_B} = \frac{1}{T_T}}$$

$$\frac{1}{3} + \frac{1}{x} = \frac{1}{1}$$

$$\left(\frac{x}{x}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{3}\right)\left(\frac{1}{x}\right) = 1$$

$$\frac{x}{3x} + \frac{3}{3x} = 1$$

$$\frac{x+3}{3x} = 1$$

$$x + 3 = 3x$$

$$3 = 2x$$

$$\boxed{1.5\ hours = x}$$

- It takes 1.5 hours for Tim to mow the lawn. Linda can mow the same lawn in 2 hours. How long will it take John and Linda, work together, to mow the lawn?
- Let t be the time for John and Linda to mow the Lawn.
- The work done by John alone is $t \times (1 / 1.5)$
  The work done by Linda alone is given by
  $t \times (1 / 2)$
- When the two work together, their work will be added. Hence
  $t \times (1 / 1.5) + t * (1 / 2) = 1$
- Multiply all terms by 6
- $6 (t \times (1 / 1.5) + t \times (1 / 2)) = 6$
- Simplify $4 t + 3 t = 6$
- Solve for t
- $t = 6 / 7$ hours = 51.5 min.

- Emma needs 6 hours to type a report and Fred needs 10 hours to type the same report.

- Emma typed a portion of the report for a few hours. Then, Frank finished typing the report.

- They, together, worked a total of 7 hours. How many hours did Emma work on this report?

- Provide your answer as a fraction in lowest terms.

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