ELMACON Preparation Session 3

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Contents

These are HARD questions. They come from the end of the sprint and target for grade 7 students. Don't worry if you struggle at first, just re-read the answers and try to pick out the pieces to build up your understanding.

Handout #1

Question 1

The sum of Abe's and Bob's heights is $220cm$ (centimetres). Abe's height minus one third of Bob's height is half of Abe's height. What is Bob's height (in cm)?

You can build two equations to solve this

1. $a + b = 220$
2. $a - b/3 = a/2$

Multiply equation 2 by the lcm to get rid of the fractions

- $6a - 2b = 3a$ or, rearranging $3a = 2b$

Now multiply both sides of equation 1 by 3

- $3a + 3b = 660$

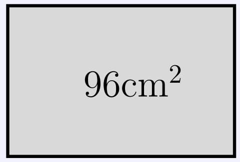
Now we can use $3a = 2b$ to get an equation only involving b

- $2b + 3b = 5b = 660$

So $b = 660/5 = 132cm$

Question 2

A rectangle has perimeter 40cm and area 96 cm^2. What is the length of the longest side of the rectangle (in cm)?



We can also solve this with equations. Let's call the long side of the shape l and the short side h. Then we can write down the area and the perimeter (remember the factors of 2!)

1. Area: $A = l \times h = 96\,cm^2$
2. Circumference $C = 2l + 2h = 40$

Since we want to know l, rearrange equation 2, so that we can replace h

- $h = 20 - l$

Then equation 1 becomes

- Area $= l(20 - l) = 20l - l^2 = 96$

Another way of writing this is

- $l^2 - 20l + 96 = 0$

In this form, we can "complete the square" to get the answer we need. We want to write down something like $(l - n)^2$, where n is some number. If you try expanding this, you'll notice that to get $-20l$ we want $n = -10$. So the square term will look like $(l - 10)^2$. Expanding that you would get $l^2 - 20l + 100$ but our equation is $l^2 - 20l + 96$ so we have to subtract 4

- $l^2 - 20l + 96 = (l - 10)^2 - 4 = 0$

Now we can rearrange the last part to get the answer

- $(l - 10)^2 = 4$ so $l - 10 = \pm 2$ or $l = 8 or 12$

The question asks for the largest value (always read the question carefully!) so $l = 12$

Question 3

How many positive whole numbers n satisfy $n = m^k$ where m and k are also positive whole numbers, $k < 5$, and the number of digits of n is k?

- Saying that $k < 5$ and that k is a positive whole number only leaves 4 possibilities: $k = 1, 2, 3$ or 4. We can examine each of these in turn.
- For $k = 1$, the question is asking, how many positive whole numbers with 1 digit satisfy $n = m$ for some positive whole number m? All 9 positive whole numbers!
- For $k = 2$, the question is, "how many 2 digits n's satisfy $n = m^2$ for some positive whole number m? $10^2 = 100$ is too many digits, but $9^2 = 81$ works, and so will some smaller whole numbers, but $3^2 = 9$ is too few digits. For $k = 2$, $4, 5, 6, 7, 8$ and 9 work so that is 6 values.
- The pattern from $k = 2$ will apply to $k = 3$ and $k = 4$: 9^k will work, but some of the smaller numbers will not. For $k = 3$, it turns out that $5^3 = 125$ is the smallest, and for $k = 4$, $6^4 = 1296$ is the smallest.
- Last re-read the question and make sure you answer the question that was asked. There are 9 cases for $k = 1$, 6 for $k = 2$, 5 for $k = 3$ and 4 for $k = 5$. The answer is $9 + 6 + 5 + 4 = 24$.

Question 4

There are 15 students in a judo class, 12 have a blue belt and 3 have a black belt. The students are split into 3 groups of 5 students to practice. Every group must have 4 blue belt students and 1 black belt student. How many ways are there to split the class into these groups?

Straight away we can tell this is going to be about picking combinations, but the hard part is going to be making sure we don't count groups more than once!

Start by assigning the students with the black belts. There are 3 of these. The first one can go into any of the 3 groups, the second into any of the remaining 2 and the third can only go into the last group. That means we could assign them to $3! = 3 \times 2 \times 1 = 6$ different groups. Now think about those groups, do we really have six different groups? No, they are completely defined by their membership. If you imagine labeling them groups the same counting as above would tell you that there are $3! = 6$ ways of labeling them so the total number of ways of assigning the black belts is 1!

Now think about assigning the blue belts to these groups. For the first group, we want to choose 4 people from the 12 blue belts.

$$\binom{12}{4} = \frac{12!}{4!(12-4)!} = \frac{12 \times 11 \times \cdots 1}{(4 \times 3 \times 2 \times 1)(8 \times 7 \times \cdots 1)} = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495$$

For the next group, we need to choose 4 students from 8, $\binom{8}{4} = 70$ and for the last group $\binom{4}{4} = 1$.

Putting that all together, there are

$$\frac{6}{6} \times \binom{12}{4} \times \binom{8}{4} \times \binom{4}{4} = 34650$$

possible groups

Handout #2

Question 1

Two buses leave the station once an hour: a red bus leaves at 47 minutes past the hour and a blue bus at N minutes past the hour. Lili arrives once daily at the station at a random time and boards the next bus that departs. On average, she boards the red bus 65% of the days. What is the value of N?

Imagine a clock face: Each minute of a clock face is 6° of the circle. Now think of the clock face as a pie chart with blue and red sectors. The red sector begins at some number n of minutes, and ends at 47 minutes past the hour ($47 * 6 = 282^\circ$ of the circle). The rest of the circle is coloured blue.
Now, if Lily arrives at the station when the minute hand of the clock is in the blue sector she'll get on the blue bus, and if it is in the red sector she'll get on the red bus. The percentage in the question tells us that the red sector must make up $65\% = 234^\circ$ of the circle!

$$282^\circ - 234^\circ = 48^\circ$$

So we know the clock hand is at 48° or $\frac{48}{6} = 8$ minutes!

Question 2

A new light bulb loses less than 0.5% of its brightness per year. That means it is at least 99.5% of its original brightness after one year, and at least 99.0025% of its original brightness after 2 years. If it still works after 8 years, what is its minimum possible brightness, as a percentage of its original brightness at that time? Round the answer to the nearest whole number.

You can do this question by multiplying 0.995 eight times, it will just take a while. Another strategy is to notice that $0.995 = 1 - 0.005$.

$$0.995^8 = (1-0.005)^8 = \underbrace{(1-0.005)\times(1-0.005)\cdots\times(1-0.005)}_{8\,\text{times}}$$

If you started writing this out, you would quickly notice that $0.005^2 = 0.000025$ is tiny, and higher powers will be even smaller! The only terms that are going to matter are the ones with single powers of 0.005 and if you start expanding some of those terms you'll convince yourself that there are 8 of those

$$8 \times 0.005 = 0.04$$

or 4%. That means the answer is 96%

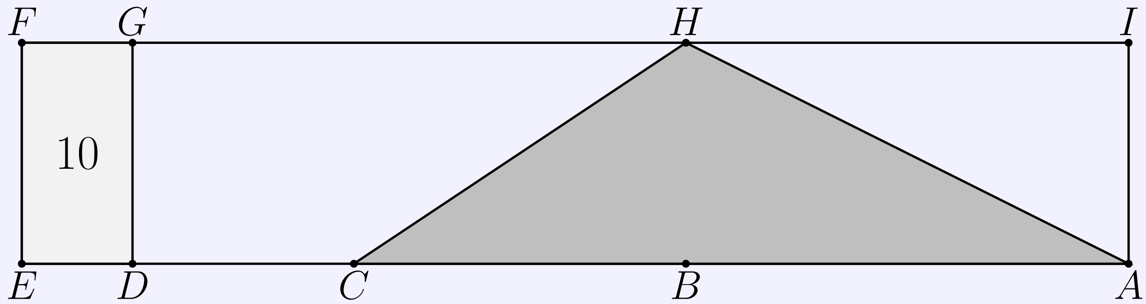
If you are comfortable with the binomial theorem, the other way to look at this is

$$(1-0.005)^8 = 1^8 - \binom{8}{1}\times 1^7 \times (0.005) + \binom{8}{2}\times(0.005)^2 + \cdots + \binom{8}{8}\times(0.005)^8$$

But using the same argument as before, only the first two terms matter. For example $\binom{8}{2} \times 0.005 = 0.0007$. We are asked to round to the nearest percent, so only the first two decimal places are going to matter.

Question 3

In the figure below $AB = 2CD$, $BC = 3DE$, $CD = 2DE$. The area of rectangle $DEFG$ is 10 cm^2. Find the area of shaded region ($\triangle ACH$) in cm^2.



Let's call $DE = x$, and lets imagine a point J directly above C on the line FI. If we can figure out the area of $AIJC$ the area of the shaded region will be half of that.
From the problem statement $CD = 2x$ and $AB = 4x$, so the base of $AIJC$ is $7x$.
Now since we know the base of $EDFG$ (x) and it's area 10, we know it's height must be $10/x$. But then we know the height of $AIJC$ must be the same so it's area is

$$7x \times \frac{10}{x} = 70$$

Again, make sure you read the question carefully and check what you were asked. For this question, we want the area of the shaded triangle which must be half of 70. The answer is 35.

Question 4

N and K are positive whole numbers. The greatest common factor of N and K is 6. The smallest common multiple is 180. What is the smallest possible value of $N + K$?

These words "least common multiple" and "greatest common factor" should be a big hint here. If N and K have 6 as their greatest common factor, then there are numbers n and k such that

$$N + K = 6n + 6k = 6(n + k)$$

Where n and k will have *no* common factors (relatively prime).
The least common multiple of N and K is then $6ab = 180$ so $ab = 30 = 5 * 3 * 2$.
The smallest possible value of their sum will be when they are closest which will be $a = 5$, $b = 6$, or $N = 30$, $K = 36$ so that $N + K = 66$.

Handout #3

Question 1

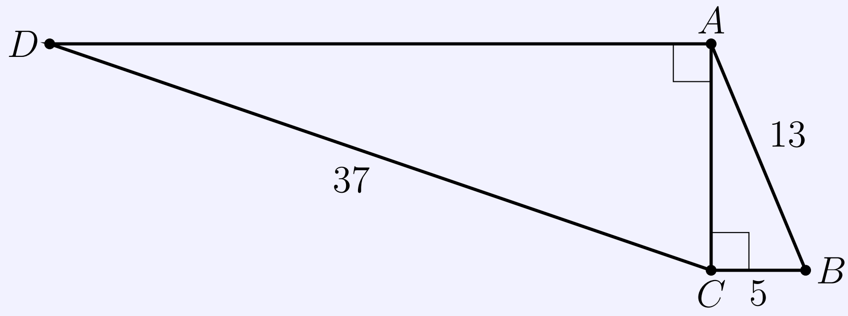
What is the smallest whole number greater than 50 that can be expressed as the sum of two different prime numbers in exactly two different ways?

One thing to notice is that you'll only have to check the even numbers greater than 50. I don't know if there is a way to do this that doesn't just exhaustively check, but

- $52 = 23 + 29 = 5 + 47 = 11 + 41$
- $54 = 5 + 49 = 7 + 47 = 11 + 43$
- $56 = 3 + 53 = 13 + 43 = 19 + 37$
- $58 = 5 + 53 = 11 + 47 = 17 + 41$
- $60 = 7 + 53 = 13 + 47 = 17 + 43$
- $62 = 3 + 59 = 19 + 43$ WINNER!

Question 2

In the figure below, $AB = 13$, $BC = 5$, AC perpendicular to both BC and AD, and $CD = 37$. What is the area of the polygon $ABCD$?



This is mostly just Pythagoras theorem.

- $AC^2 = 13^2 - 5^2 = 144 = 12^2$
- $AD^2 = 37^2 - AC^2 = 1369 - 1444 = 1225 = 35^2$

Then the area of the big triangle is

- $\frac{1}{2} \times 12 \times 35 = 210$

And the small one is

- $\frac{1}{2} \times 12 \times 5 = 30$

The question wants the sum of these, which is $210 + 30 = 240$.

Question 3

1000 runners competed in a very long race and 82% did not finish. 30% of the runners aged 20 or older finished the race and they made up 75% of all the people who finished. How many participants were less than 20 years old?

This question has very tricky wording. I'm going to refer to young and old runners.
First, how many people finished the race? $(1 - \frac{82}{100}) \times 1000 = 180$ runners finished.
Next, how many of them were old? $180 \times \frac{3}{4} = 135$
Now if that was only 30% of the old people how many old people were there in total? $135 \times \frac{100}{30} = 450$.
*(30% is $\frac{30}{100}$ and we want to **divide** by 30%, so we multiply by $\frac{100}{30}$)*
Now we know the total number of runners and the total number of old runners, so the number of young runners must be $1000 - 450 = 550$.

Question 4

How many non-congruent triangles with whole number sides have a perimeter of 17?

This question is about using the triangle inequality. a and b are the short sides and c is the hypotenuse of a triangle; the triangle inequality says $a + b > c$.

From the question we have $a + b + c = 17$. If you replace $a + b$ in this we get $2c < 17$. If that is true, then $2c < 18$ or $c < 9$. Now we can look at cases. Without losing anything we can assume that $a \leq b$ (otherwise we're just relabeling the same triangles). In all of the cases $b < c$.

- For $c = 8$, we must have $a + b = 9$. The only ways we can do that are
  - $a = 1, b = 8$
  - $a = 2, b = 7$
  - $a = 3, b = 6$
  - $a = 4, b = 5$
- For $c = 7$, we must have $a + b = 10$.
  - $a = 3, b = 7$
  - $a = 4, b = 6$
  - $a = 5, b = 5$
- For $c = 6$, we must have $a + b = 11$.
  - $a = 5, b = 6$

Counting those up we have 8 triangles.

Handout #4

Question 1

A bag contains 8 red balls, 6 blue balls and 6 green balls. Three balls are selected at random, without replacement. If the first ball was red, what is the probability that all of the selected balls will be different colours? Express the answer as a fraction in lowest terms.

The red ball has already been drawn, so we are only asking about drawing one blue and one green ball from what remains.

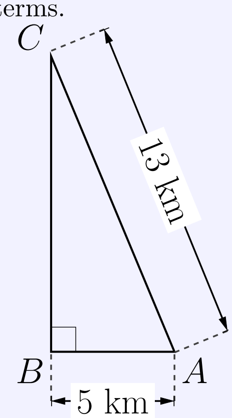
There are $19 \times 18 = 342$ ways of choosing any two balls. That will give us the denominator

There are 6×6 ways of pulling the blue then the green and 6×6 ways of pulling the green then the blue.

$$\frac{36 + 36}{342} = \frac{4}{19}$$

Question 2

$\triangle ABC$ is a right triangle ($\angle ABC = 90°$). $AB = 5\ km$, $AC = 13\ km$. Ann and Bob start travelling from point A at the same time. Ann travels from point A to point C along the line AC at a speed of 4 $\frac{km}{h}$. Bob travels from point A to point B along line AB at a speed of 6 $\frac{km}{h}$ and then continues, without stopping, to point C along line BC at a constant speed V (in $\frac{km}{h}$). Ann and Bob arrive at point C at the same time. What is the value of V in $\frac{km}{h}$? Express the answer as a fraction in lowest terms.



It helps to know

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

AND that you can rearrange that equation, e.g.

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

First we have to do a little bit of pythagoras to figure out that $BC^2 = 13^2 - 5^2 = 144 = 12^2$, so $BC = 12\,km$.

Next we can write down the time it took for ann to go from A to C

$$T_{Ann} = \frac{AC}{v_{AC}} = \frac{13}{4}$$

Next we can do the same thing for Bob, but his time is the sum of the two legs

$$T_{Bob} = \frac{AB}{v_{AB}} + \frac{BC}{v} = \frac{5}{6} + \frac{12}{v}$$

Now we know that $T_{Bob} = T_{Ann}$, so we just need to multiply away those fractions

$$\frac{13}{4} = \frac{5}{6} + \frac{12}{v} \iff 39 = 10 + \frac{144}{v}$$

Rearanging gives $v = \frac{29}{144}\,km/h$.

Question 3

Taylor receives an award for outstanding achievements in mathematics. She deposits 40% of the award in her bank account, and uses $\frac{7}{12}$ of what remains to cover her travel expenses. She donates the rest of the money to charity, giving $\frac{2}{3}$ of that to United Way and the remaining \$750 to the Red Cross. How big was the initial award (in dollars)?

This is another very tricky word problem. I find it helpful to think about her receiving x dollars in total.

- 40% goes to the bank, so she keeps $0.6x$
- She spends $\frac{7}{12}$ of that, which leaves $\frac{5}{12} \times 0.6x$.
- She donates $\frac{2}{3}$ of what's left to United Way and *the remaining* \$750 to the Red Cross

Think about the last amount. If she gave 2/3 of what was left to united way, the remaining 750 was 1/3 of what was left, so what was left was

$$x \times \frac{5}{12} \times \frac{6}{10} \times \frac{1}{3} = 750$$

Solving this we get $\frac{30x}{360} = 750$ or $x = 9000$.

Question 4

The total value of all the coins in a bag is \$85.00. There are equal numbers of each of the following coin values: \$0.05, \$0.10, \$0.25, \$1.00, \$2.00. There are no coins of any other values in the bag. How many coins are there in the bag?

I'm not sure if there is a neat way of doing this, but what the question basically says is

$$n(0.05 + 0.10 + 0.25 + 1.00 + 2.00) = 3.40 \times n = 85.00$$

It's usually easier to deal with integers so $n = \frac{850}{34} = 25$, and there are 5 coins so the answer is 125.

Handout #5

Question 1

Bob lives in Burnaby and invites some of his friends to a party. 17 of his friends are from Vancouver and 21 are from Richmond. Each guest shakes hands with all of the other people from the same city as them (once), and each guest shakes hands with Bob (once). How many handshakes are there in total?

Remember that me shaking your hand is the same as you shaking my hand, so be careful to avoid double counting.

- There are 17 people from Vancouver, so there are $17 \times (17-1) = 272$ ways of making those pairs, but as above, we have to divide by 2, giving 136.
- There are 21 from Richmond, so $21 \times (21-1) = 420$ ways of making those pairs, but as above, we have to divide by 2, giving 210.
- Finally everyone shakes hands with Bob, which is $17 + 21 = 38$ more handshakes

In total then $136 + 210 + 38 = 384$.

Question 2

A 2-digit whole number is randomly selected. If the digit sum of this number is 9, what is the probability that the number is less than 80? Express the answer as a fraction in lowest terms.

I don't know if there is a better way to do this, but the pairs which digit sum to 9 are

- (1,8)
- (2,7)
- (3,6)
- (4,5)
- (5,4)
- (6,3)
- (7,2)
- (8,1)
- (9,0)

and of those 9, 7 would be less than 80. So $\frac{7}{9}$.

Question 3

Given that $p_1 \times p_2 \times \cdots \times p_N = 2025$ where $p_1, p_2, \ldots, p_N$ are all primes, what is the value of $p_1 + p_2 + \cdots + p_N$?

It is always a good idea to know the prime decomposition of the year of your ELMACON contest!

$$2025 = 3^4 \times 5^2$$

Then $3 + 3 + 3 + 3 + 5 + 5 = 22$

Question 4

What is the sum of the 3 smallest whole numbers that have exactly 18 factors?

Hints: 1) The value of each of these 3 numbers is less than 300. 2) The 3 smallest whole numbers that have exactly 2 factors are 2, 3, 5 — so their sum is 10.